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DEVELOPMENT OF THE AVERAGE LIKELIHOOD FUNCTION FOR CODE DIVISION MULTIPLE ACCESS (CDMA) USING BPSK AND QPSK SYMBOLS

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FOR THE DIRECTOR:

/ S /
JAMES PERRETTA, Acting Chief
Cyber Operations Branch

/ S /
WARREN H. DEBANY, JR.
Technical Advisor, Information
Exploitation & Operations Division
Information Directorate

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Alfredo Vega Irizarry

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14. ABSTRACT

This interim report summarizes the most significant findings during the first year of research under the in-house effort titled "Average Likelihood Methods for CDMA". In this report, we extend the derivation of the average likelihood function for three additional CDMA transmission models. These are: QPSK-Code/BPSK-Data, BPSK-Code/QPSK-Data and QPSK-Code/QPSK-Data. All the new developments can be expressed in the same form of the BPSK-Code/BPSK-Data presented in AFRL-RI-RS-TR-2014-093.

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Summary

This interim report summarizes the most relevant findings during the first year of research on the effort titled Average Likelihood Methods for Classification of CDMA. The goal of the research is to present the development of several analytical form of the average likelihood for CDMA using combinations of BPSK and QPSK symbols. In a previous report, the case of CDMA using a BPSK code matrix and a BPSK data vector was studied and published in Technical Report AFRL-RI-RS-TR-2014-093. In this report, the average likelihood for different CDMA transmission models will be developed using similar definitions and propositions.

Decision theoretic methods based on average likelihood are the only methods that guarantee the best classification in noise according to the classical Detection Theory. Currently there is a lack of average likelihood methods for CDMA. This research has the purpose to establish a foundation for new classification and estimation of CDMA signals.

Keywords: DS/CDMA signals, BPSK, QPSK, average likelihood function, classification, estimation.

1. Introduction

The characterization of signals in terms of the modulation parameters is a field of interest relevant to military and software defined radio. Early modulation classification algorithms were based on ad-hoc methods. These methods were applied for classifying analog signals such as AM and FM. The methods were extended to digital types such as BPSK and QPSK. A collection of these algorithms can be found in [1]. The main benefit of these Ad-Hoc algorithms is in providing classification rules that are easy to implement. The mayor disadvantage is their inability to guarantee any degree of performance in noise.

But the field of modulation classification underwent a significant change of direction with Kim, Huang and Polydoros papers [2] and [3]. These papers applied classical detection theory to MPSK signals. Starting from a model, these authors developed an average likelihood approximation for MPSK signals. The authors also demonstrated that some of the heuristic classification rules used to classify signals were special cases of the Average Likelihood Ratio Test (ALRT). With these findings, following authors began the development of likelihood methods for a combination of QAM and MPSK methods as shown in Table 1.

Table 1 was presented in a 2007 conference paper [4]. As we can see, there is a lack of decision theoretic methods for CDMA signals. In 2012, the same authors published a paper on the classification of CDMA signals based on its code length [5]. The approach consists on extracting cyclostationary features, i.e., peaks in the spectral correlation function, and then training a neural network. Feature based approaches started to emerge as an alternative of Decision Theoretic methods with increase complexity. As in the case of Ad-Hoc methods, feature-based approaches do not guarantee performance and the main weakness of this method is the dependence of the classification on the training set. In addition, the proposed method for CDMA signals can process signals for a limited set of code lengths due to its inherent complexity.

Developing decision theoretic approaches for CDMA signals is a challenging task due to the large number of unknown variables that require averaging. Reference [6] shows that it is possible to express the likelihood in a compact form and apply further simplifications. The detection rules were applied the classification of signals up to code lengths of 2^{12}.

During the second year of performance of this AFOSR funded effort, the research explored the extension of average likelihood for CDMA signals constructed from a combination of BPSK and QPSK symbols. The research found that it was possible to develop average likelihood equations using previous findings.

The usage of the average likelihood applied to CDMA is not limited to the problem of classification. A more ambitious goal is to be able to estimate the spreading matrix and multi-user detection. These problems are considered for future work.

Table 1 Summary of Likelihood-Based Classifiers

Authors	**Classifiers**	**Modulations**	**Unknown Parameters**	**Channel**
Sills	ALRT	BPSK, QPSK, 16QAM, V29, 32QAM, 64QAM	Carrier phase	AWGN
Wei and Mendel	ALRT	16QAM, V29	-	AWGN
Kim and Polydoros	Quasi-ALRT	BPSK, QPSK	Carrier phase	AWGN
Huang and Polydoros	Quasi-ALRT	UW, BPSK, QPSK, 8PSK, 16PSK	Carrier phase, timing offset	AWGN
Sapiano and Martin	ALRT	UW, BPSK, QPSK, 8PSK	-	AWGN
Long	Quasi-ALRT	16PSK, 16QAM, V29	Carrier phase	AWGN
Hong, Ho	ALRT	BPSK, QPSK	Symbol level	AWGN
Beidas and Weber	Quasi-ALRT	32FSK, 64FSK	Phase jitter	AWGN
Beidas and Weber	Quasi-ALRT	32FSK, 64FSK	Phase jitter, timing	AWGN
Panagiotu	GLRT,HLRT	16PSK, 16QAM, V29	Carrier phase	AWGN
Chugg	HLRT	BPSK, QPSK, OQPSK	Carrier phase, signal power, PSD	AWGN
Hong and Ho		BPSK, QPSK	Signal Level	AWGN
Hong and Ho	HLRT	BPSK, QPSK	Angle of Arrival	AWGN
Dobre	HLRT	BPSK, QPSK, 16QAM, V29, 32QAM, 64QAM	Channel amplitude, phase	Flat Fading
Abdi	ALRT, quasi-HLRT	16QAM, 32QAM, 64QAM	Channel amplitude and phase	Flat Fading

2. Approach

The approach for deriving the average likelihood function for CDMA consisted on expressing the average likelihood in a compact form. The function can be expressed as a sum of product of hyperbolic cosine functions. Further simplification can be achieved by implementing a hard decision classifier. In this approach, the variance of the probability of the code matrix is reduced to a minimum value. This is equivalent to classifying the CDMA signal using the most probable code for a given code length.

The accomplished work on BPSK codes and BPSK data considered the case of chip synchronous signals with no knowledge on the beginning of the sequence. Both balance and unbalanced load cases were considered under the synchronization assumption. A similar methodology will be applied to general CDMA models using a combination of BPSK and QPSK symbols. The following CDMA transmission models were investigated.

- QPSK code – BPSK data,
- BPSK code – QPSK data, and
- QPSK code – QPSK data.

3. Development of the Average Likelihood

3.1 General Model

The goal is to classify all types of CDMA signals by their code length L and number of users U in the presence of Additive White Gaussian Noise (AWGN). A CDMA signal $\vec{x}$ of size $L^\star \times 1$ is generated by modulating a spreading code matrix C of size $L^\star \times U^\star$, with the information symbol vector $\vec{b}$ of size $U^\star \times 1$. Both, matrix C and vector $\vec{b}$ are constructed from a combination of BPSK $\{\pm 1\}$ and QPSK $\{\pm 1, \pm i\}$ symbols.

For simplicity, we assume that the energy per symbol per user E is constant, i.e., a balanced load. The code length and number of active users is assumed to be unknown in the model; therefore, instead of using the true dimensions $L^\star$ and $U^\star$, the model uses the hypothetical dimensions L and U as follows:

$$\vec{x} = \sqrt{E/L}\, C\, \vec{b} \tag{1}$$

The spreading matrix is a block matrix composed of column vectors $\vec{c}_{*,i}$ with low cross-correlation. A metric that characterizes the spreading matrix is known as the Total Square Correlation (TSC). A modified TSC is defined in this paper by [7]

$$\tau(C) \triangleq \sum_{i=0}^{U-1} \sum_{\substack{j=0 \\ j\neq i}}^{U-1} \left|\langle \vec{c}_{*,i}, \vec{c}_{*,j} \rangle\right|^2 \tag{2}$$

An alternative definition of (2) for a code matrix with either BPSK or QPSK symbols can be expressed in terms of the Frobenius Norm with L is the number of rows of $C^H C$ and U is the rank of $C^H C$:

$$\tau(C) \triangleq \|C^H C\|_F^2 - L^2 \cdot U \tag{3}$$

The noise vector $\vec{n}$ has zero mean, and covariance matrix $\Sigma = N_0/2 \cdot I$, where I is an $L \times L$ identity matrix. The receiver has no knowledge of the spreading code, the code length or the number of users. For a given hypothesis $\mathcal{H} = \{L, U\}$, the receiver makes the assumption that the code length is L without necessarily implying that $L = L^{\star}$. The channel is modeled by:

$$\vec{y}_{\epsilon} = \vec{x}_{\epsilon} + \vec{n} \tag{4}$$

where $\vec{y}_{\epsilon}$ represents the received vector delayed by ϵ number of chips.

The likelihood is conditioned on the hypothesis $\mathcal{H} = \{L, U\}$, the spreading matrix C, and the information vector $\vec{b}$ .The delay $\vec{x}_{\epsilon}$ is unknown to the receiver; therefore, the comparison is done with respect to $\vec{x}$ from a hypothetical model. Under the assumption of no frame synchronization, our conditional likelihood is given by:

$$\lambda\left(\vec{y} | \mathcal{H} = \{L, U\}, C, \vec{b}, \epsilon\right) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{(\vec{y}_{\epsilon} - \vec{x})^H \Sigma^{-1} (\vec{y}_{\epsilon} - \vec{x})}{2}} \tag{5}$$

3.2 Average Likelihood Function

The construction of our likelihood function requires averaging over the variables $C, \vec{b}$ and ϵ. The variable ϵ will be ignored in this development due to its lack of mathematical interest. For convenience, we define the energy-to-noise ratio per chip γ, the unnormalized signal $\vec{s}$, and the correlator output $\vec{r}$.

$$\gamma = \frac{E}{N_0 L}, \;\; \vec{S} = C\,\vec{b}, \;\; \vec{R} = \frac{\vec{y}}{\sqrt{N_0/2}} \tag{6}$$

The conditional likelihood function of (5) is a function of the correlator equation $\vec{R}^H\vec{S} + \vec{S}^H\vec{R}$ and the energy of the signal $\vec{S}^H\vec{S}$ as follows:

$$\lambda\left(\vec{r}|\mathcal{H}, C, \vec{b}\right) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{1/2} e^{\sqrt{\gamma/2}\left(\vec{R}^H\vec{S}+\vec{S}^H\vec{R}\right)-\gamma\,\vec{S}^H\vec{S}}$$

(7)

The development is shown in Appendix A. In Detection Theory, the constant term $\langle\vec{R},\vec{R}\rangle$ is usually omitted; however, the hypothesis depends on the code length and the receive vector. Caution should be exercise before trying to eliminate terms that depend on $\vec{R}$.

It was verified that a classification rule developed under the assumption of a uniformly distributed probability of the code does not work. A key observation is that there is a dependency between the elements of the spreading matrix. It is unlikely that a random selection of matrix elements would result in a low correlation matrix. Therefore, we must selectively choose the spreading matrices based on some code-design metric. The proposed approach constructs a discrete probability distribution based on our definition of the TSC. Our first choice is to propose a discrete probability that present similarities with a Gaussian distribution:

$$P(C) \triangleq \frac{e^{-\beta\cdot\tau^2(C)}w(C)}{\sum_{C'} e^{-\beta\cdot\tau^2(C')}w(C')},$$

$$w(C) = \begin{cases} 0 & rank(C) \neq U \\ 1 & otherwise \end{cases}$$

(8)

The function $w(C)$ is a weight that eliminates inadmissible spreading matrices, i.e., matrices with linearly dependent codes. For perfectly uncorrelated codes $C^\star$, the TSC metric is zero and the probability achieves a maximum value. Due to the invariance of the TSC to permutation matrices, any code matrix $Q^\star$ that is a permutation of $C^\star$ will have the same probability. In the coding theory, $Q^\star$ and $C^\star$ are referred equivalent matrices [8].

The parameter β is called the precision of the distribution in Machine Learning Theory. It is defined as half the inverse of the variance of a Gaussian distribution [9]. In the case of a Gaussian distribution, the precision reshapes the probability by making it steeper. At the same time, it also affects the classification boundaries, by changing the steepness, thus converting the classification from a soft decision to a hard decision problem. Averaging over the code coefficients eliminates the dependency of the likelihood on the spreading matrix as follows:

$$\lambda(\vec{r}|\mathcal{H},\vec{b}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{1/2} \sum_C P(C)\, e^{\sqrt{\gamma/2}(\vec{R}^H\vec{S}+\vec{S}^H\vec{R})-\gamma\,\vec{S}^H\vec{S}} \tag{9}$$

Finally, the dependency of the conditional likelihood on $\vec{b}$ is removed by averaging over uniformly distributed information symbols as given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{1/2} \sum_{\vec{b}} P(\vec{b}) \sum_C P(C)\, e^{\sqrt{\gamma/2}\,(\vec{R}^H\vec{S}+\vec{S}^H\vec{R})-\gamma\,\vec{S}^H\vec{S}} \tag{10}$$

The probability $P(\vec{b})$ depends on the domain of $\vec{b}$ with $P(\vec{b}) = 2^{-U}$ for BPSK symbols and $P(\vec{b}) = 2^{-2U}$ for QPSK symbols. The order in which the expectation is taken does not affect the final result. In our study, the average over the information vector is done first for convenience as we will see later in the discussion.

3.3 Key Definitions and Propositions

The inclusion of complex variables increases the complexity of the problem by the number of unknown variables when compared to the BPSK code/BPSK data CDMA model. In order to average the conditional likelihood, we provide the following key definitions and propositions.

Definition: The set $\mathcal{S}_{\vec{a}}$ is the subset of real antipodal matrices $C = \{c_{i,j}\}^{L\times U}$ with:

$$a_i = \sum_{j=0}^{U-1} c_{i,j} \geq 0 \tag{11}$$

Definition: $\mathcal{S}_{\vec{a},t}$ is the subset of real antipodal matrices $C = \{c_{i,j}\}^{L\times U} \in \mathcal{S}_{\vec{a}}$ with:

$$\tau^2(C) = t \tag{12}$$

Definition: $\mathcal{S}_{\vec{a}_+,\vec{a}_-,t}$ is the subset of pair of matrices (Z_+, Z_-) with $Z_+ = \{\pm 1\}^{L\times U}$ and $Z_- = \{\pm 1\}^{L\times U}$ with:

$$(\vec{a}_+)_i = \sum_{j=0}^{U-1} (Z_+)_{i,j} \geq 0$$

$$(\vec{a}_-)_i = \sum_{j=0}^{U-1} (Z_-)_{i,j} \geq 0$$

$$\tau^2 \left(\frac{(Z_+ + Z_-)}{2} + \mathbb{i} \frac{(Z_+ - Z_-)}{2} \right) = t \tag{13}$$

Proposition 1: The product of a diagonal matrix $G_{L\times L} = diag(\vec{g}_{L\times 1})$ and a vector $\vec{r}_{L\times 1}$ commutes as follows:

$$G \cdot \vec{r} = diag(\vec{r}) \cdot \vec{g} \tag{14}$$

Proposition 2: The product of the block matrix

$$G = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix}$$

with diagonal matrices $G_R = diag(\vec{g}_R)$ and $G_I = diag(\vec{g}_I)$ and a block vector $\vec{R}_{2L\times 1}$ with entries

$$\vec{R}_{2L\times 1} = \begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix}$$

commutates as follows:

$$G\,\vec{R} = \begin{bmatrix} diag(\vec{r}_I) & diag(\vec{r}_R) \\ -diag(\vec{r}_R) & diag(\vec{r}_I) \end{bmatrix} \begin{bmatrix} \vec{g}_R + \vec{g}_I \\ \vec{g}_R - \vec{g}_I \end{bmatrix} \tag{15}$$

Proposition 3: The average of a function $f\left(C\cdot\vec{b}\right)$ over C and $\vec{b}$ where $C\in\{\pm1\}^{L\times U}$ and $\vec{b}\in\{\pm1\}^{U\times 1}$ is given by:

$$\frac{1}{2^{L\cdot U+U}}\sum_{C}\sum_{\vec{b}} f\left(C\cdot\vec{b}\right)=\frac{1}{2^{L\cdot U}}\sum_{C} f\left(C\cdot\vec{1}\right) \tag{16}$$

Proposition 4: Let $\vec{b}=\vec{b}_R+\mathbb{i}\,\vec{b}_I$ with $\vec{b}\in\{\pm1,\pm\mathbb{i}\}^{U\times 1}$, $\vec{b}_R=Re\{\vec{b}\}$ and $\vec{b}_I=Im\{\vec{b}\}$. Also, let $\vec{b}_+\in\{\pm1\}^{U\times1}$ and $\vec{b}_-\in\{\pm1\}^{U\times1}$, then

$$\frac{1}{2^{2\cdot U}}\sum_{\vec{b}_R}\sum_{\vec{b}_I} f\left(\vec{\alpha}^T\left(\vec{b}_R+\vec{b}_I\right)\right) f\left(\vec{\beta}^T\left(\vec{b}_R-\vec{b}_I\right)\right)=\frac{1}{2^U}\sum_{\vec{b}_+} f\left(2\vec{\alpha}^T\vec{b}_+\right)\frac{1}{2^U}\sum_{\vec{b}_-} f\left(2\vec{\beta}^T\vec{b}_-\right) \tag{17}$$

The previous proposition states that for averaging purposes, the product $\vec{b}_+$ and $\vec{b}_-$ acts as two independent antipodal vectors.

Proposition 5: The average of the function $e^{\vec{\alpha}^T\vec{g}}$ over $\vec{g}$, where $\vec{g}=\{\pm1\}^{L\times1}$, is given by:

$$\frac{1}{2^L}\sum_{\vec{g}=\{\pm1\}^{L\times1}} e^{\vec{\alpha}^T\vec{g}}=\prod_{i=0}^{L-1}\cosh(\alpha_i) \tag{18}$$

Proposition 6: The TSC is invariant to permutations and sign-inversions of rows and columns.

Proposition 7: The limiting case as $\beta\to\infty$ of a rational exponential function with $\tau_i^2\geq 0$, $d_i\geq 0$, $f_i>0$ is given by:

$$\lim_{\beta\to\infty}\frac{\sum_{i=0}^{N} d_i e^{-\beta\cdot\tau_i^2}}{\sum_{i=0}^{N} f_i e^{-\beta\cdot\tau_i^2}}=\frac{d_0}{f_0} \tag{19}$$

The proofs of these propositions are found in Appendix C.

3.4 Development of a Compact Average Likelihood

For each of the three CDMA transmission models in section, we need to find suitable models that use a development similar to the already solved case, i.e., the BPSK-code/BPSK-data model.

The development requires computing the correlation term $\left(\vec{R}^H\vec{S} + \vec{S}^H\vec{R}\right)$, the energy term $\vec{S}^H\vec{S}$ of the conditional likelihood of (10) and the covariance matrix of each CDMA model. The covariance matrix is uncorrelated with a common variance σ^2 for all the variables.

3.4.1 Revisiting BPSK-Code/BPSK-Data Model

The model for a BPSK-Code/BPSK-Data transmission model is given by:

$$\begin{bmatrix}\vec{r}_R\\ \vec{r}_I\end{bmatrix} = \begin{bmatrix}G_R Q_R\\ \mathbf{0}\end{bmatrix}\vec{b}_R, + \begin{bmatrix}\vec{n}_R\\ \vec{n}_I\end{bmatrix}$$

$$\begin{aligned} G_R &= diag(\vec{g}_R)\\ \vec{g}_R &\in \{\pm 1\}^{L\times 1}\\ \vec{b}_R &\in \{\pm 1\}^{U\times 1}\\ Q_R &\in \{\pm 1\}^{L\times U} \end{aligned} \tag{20}$$

or equivalently,

$$\vec{R} = \begin{bmatrix}\vec{r}_R\\ \vec{r}_I\end{bmatrix}, \qquad \vec{S} = \begin{bmatrix}G_R & \mathbf{0}\\ \mathbf{0} & G_R\end{bmatrix}\begin{bmatrix}Q_R & \mathbf{0}\\ \mathbf{0} & Q_R\end{bmatrix}\begin{bmatrix}\vec{b}_R\\ \vec{0}\end{bmatrix}, \qquad \vec{N} = \begin{bmatrix}\vec{n}_R\\ \vec{n}_I\end{bmatrix}$$
$$\vec{R} = \vec{S} + \vec{N} \tag{21}$$

The imaginary part of (21) does not provide any useful information and vanishes in the correlator equation. By using Proposition 1, the correlation expression becomes:

$$\vec{R}^T \vec{S} = \vec{g}_R \, D \, Q_R \, \vec{b}_R$$

$$D = diag(\vec{r}_R) \tag{22}$$

Also, we compute the energy term, which is given by:

$$\vec{S}^T \vec{S} = \vec{b}_R^T \, Q_R^T Q_R \, \vec{b}_R$$

$$\vec{S}^T \vec{S} = \vec{1}^T Z^T Z \, \vec{1} \tag{23}$$

with $Z = Q_R \, diag\left(\vec{b}_R\right)$. The covariance matrix for complex signals is uncorrelated as shown below:

$$\mathbb{E}\left\{\vec{N}\vec{N}^H\right\} = \frac{N_0}{2} \, I_{L\times L} \tag{24}$$

The previous equations are substituted in (10). The average likelihood in its raw form is given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle \vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\vec{g}_R} P(\vec{g}_R) \sum_{Z} P(Z) \sum_{\vec{b}} P\left(\vec{b}_R\right) e^{\sqrt{2\gamma}\,(\vec{g}_R \, D \, Z \, \vec{1}) - \gamma \|Z \, \vec{1}\|^2} \tag{25}$$

with $P(\vec{g}_R) = 1/2^L$. By splitting the matrix C as the product of $G \, Q$, we have slightly modified the definition of $P(Q)$ or $P(Z)$. Averaging over $\vec{g}_R = diag(G)$ doubles the number of possible matrices Z whenever a row of the product $Z \, \vec{1}$ is zero, i.e., multiplying zero by +1 is not different from multiplying zero by -1. We define $P(Z)$ as:

$$P(Z) \triangleq \frac{e^{-\beta\cdot\tau^2(Z)}\ 2^{-\sum_i Count\left(\left(Z\ \vec{1}\right)_i==0\right)}\ w(Z)}{\sum_{Q'} e^{-\beta\cdot\tau^2(Q')}\ 2^{-\sum_i Count\left(\left(Z'\ \vec{1}\right)_i==0\right)}\ w(Z')},$$

$$w(Z) = \begin{cases} 0 & rank(Z) \neq U \\ 1 & otherwise \end{cases}$$

(26)

The average over $\vec{b}_R$ is simplified by applying Proposition 3.

$$\sum_{\vec{b}_R} P\left(\vec{b}_R\right) e^{\sqrt{2\gamma}\ (\vec{g}_R\ D\ \ Q_R\ \vec{b}_R) - \gamma\ \vec{b}_R^T\ Q_R^T Q_R\ \vec{b}_R} = e^{\sqrt{2\gamma}\ (\vec{g}_R\ D\ \ Z\ \vec{1}) - \gamma\ \left\|Z\ \ \vec{1}\right\|^2}$$

(27)

The average over $\vec{g}_R$ is simplified by applying Proposition 5.

$$\sum_{\vec{g}_R} e^{\sqrt{2\gamma}\ (\vec{g}_R\ D\ Z\ \vec{1}) - \gamma\ \left\|Z\ \ \vec{1}\right\|^2} = e^{-\gamma\ \left\|Z\ \ \vec{1}\right\|^2} \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma}\ (D\ \vec{a})_i\right)$$

(28)

The summation over Z is split in the summation over all possible $\vec{a}$ formed from $\vec{a} = Z\ \vec{1}$ with $Z \in\ \mathcal{S}_{\vec{a},t}$, all possible TSC terms and all matrices elements of the set $\mathcal{S}_{\vec{a},t}$. Matrices in the set $\mathcal{S}_{\vec{a},t}$ have the same the energy term and TSC according to Proposition 7.

$$\sum_{Z} \dots = \sum_{\vec{a}} \sum_{t} \sum_{Z \in\ \mathcal{S}_{\vec{a},t}} \dots$$

(29)

The resulting likelihood is a compact form given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\vec{a}} \alpha(\beta,\vec{a}) e^{-\gamma\|\vec{a}\|^2} \prod_{i=0}^{L-1} \cosh(\sqrt{2\gamma}\,(D\,\vec{a})_i)$$

$$\vec{a} = Z\,\vec{1} \tag{30}$$

where

$$\alpha(\beta,\vec{a}) = \sum_t \sum_{Z\in S_{\vec{a},t}} P(Z). \tag{31}$$

The coefficient $\alpha(\beta,\vec{a})$ in (31) implies that detecting CDMA signals using the provided stochastic model requires a characterization of all possible code matrices for a given code length. This fact can be seen as a weakness of the average likelihood approach, since the initial assumption does not include the knowledge on the code matrix. In practice, the user would need to have partial knowledge on possible codes used before using equation (30).

3.4.2 BPSK-Code/QPSK-Data Model

The model for BPSK-Code/QPSK-Data is given by:

$$\begin{bmatrix}\vec{r}_R\\ \vec{r}_I\end{bmatrix} = \begin{bmatrix}G_R & \mathbf{0}\\ \mathbf{0} & G_R\end{bmatrix}\begin{bmatrix}Q_R & \mathbf{0}\\ \mathbf{0} & Q_R\end{bmatrix}\begin{bmatrix}\vec{b}_R\\ \vec{b}_I\end{bmatrix} + \begin{bmatrix}\vec{n}_R\\ \vec{n}_I\end{bmatrix}$$

$$\begin{gathered}G_R = diag(\vec{g}_R)\\ \vec{g}_R \in \{\pm 1\}^{L\times 1}\\ \vec{b}_R + \mathbb{i}\,\vec{b}_I \in \{\pm 1, \pm\mathbb{i}\}^{U\times 1}\\ Q_R \in \{\pm 1\}^{L\times U}\end{gathered} \tag{32}$$

The model is impractical for developing a compact average likelihood function. Instead, an equivalent and more convenient model is used for calculating the average likelihood:

$$\vec{R} = \begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix}, \qquad \vec{S} = \begin{bmatrix} G_R & \mathbf{0} \\ \mathbf{0} & G_R \end{bmatrix} \begin{bmatrix} Q_R & \mathbf{0} \\ \mathbf{0} & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_R + \vec{b}_I \\ -(\vec{b}_R - \vec{b}_I) \end{bmatrix}, \qquad \vec{N} = \begin{bmatrix} \vec{n}_R + \vec{n}_I \\ -(\vec{n}_R - \vec{n}_I) \end{bmatrix}$$

$$\vec{R} = \vec{S} + \vec{N}. \tag{33}$$

The equivalent model is a linear combination of (32). In other to simplify the development, we notice from Proposition 4 that the summation over the block vector:

$$\vec{B}_{\pm} = \begin{bmatrix} \vec{b}_R + \vec{b}_I \\ -(\vec{b}_R - \vec{b}_I) \end{bmatrix} \tag{34}$$

acts as an independent variable vector with binary antipodal symbols. Also, the covariance matrix for complex signals is given by:

$$\mathbb{E}\{(\vec{n}_R + \vec{n}_I)(\vec{n}_R + \vec{n}_I)^H\} = \frac{N_0}{2}\, I_{L\times L} \tag{35}$$

Only half of the equation variables in (33) provide the necessary information for computing the average likelihood. It is possible to reuse average likelihood for BPSK-code/QPSK-data CDMA in (30) and (31) by simply replacing D with:

$$D = diag(\vec{r}_R + \vec{r}_I) \tag{36}$$

The BPSK-code/QPSK-data CDMA model splits the transmitted signal in two orthogonal components. The average likelihood recombines the real and imaginary components of $\vec{r}$ in (32) in a single real vector $\vec{r}_R + \vec{r}_I$ to form a model similar to the BPSK-code/BPSK-data model.

3.4.3 QPSK-Code/BPSK-Data Model

Similar steps can be used to derive the formulas for all CDMA cases using BPSK and QPSK symbols. The model for QPSK-Code/BPSK-Data is given by:

$$\begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix} = \begin{bmatrix} G_R & -G_I \\ G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_R \\ \vec{0} \end{bmatrix} + \begin{bmatrix} \vec{n}_R \\ \vec{n}_I \end{bmatrix}$$

$$G_R = diag(\vec{g}_R)$$
$$G_I = diag(\vec{g}_I)$$
$$\vec{g}_R + \mathbb{i}\vec{g}_I \in \{\pm 1, \pm \mathbb{i}\}^{L\times 1}$$
$$\vec{b}_R \in \{\pm 1\}^{L\times 1}$$
$$Q_R + \mathbb{i}\, Q_I \in \{\pm 1, \pm \mathbb{i}\}^{L\times U}$$

(37)

This model is also impractical for developing a compact likelihood function. Instead, the equivalent model from Appendix B should be used:

$$\vec{R} = \begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix}, \qquad \vec{S} = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R - Q_I \\ -(Q_R + Q_I) \end{bmatrix} \vec{b}_R, \qquad \vec{N} = \begin{bmatrix} \vec{n}_R + \vec{n}_I \\ -(\vec{n}_R - \vec{n}_I) \end{bmatrix}$$

$$\vec{R} = \vec{S} + \vec{N}$$

(38)

The correlator term $\vec{R}^T\vec{S}$ can be expressed in terms of two independent variables $\vec{g}_+ = \vec{g}_R + \vec{g}_I = \{\pm 1\}^{L\times 1}$ and $\vec{g}_- = \vec{g}_R - \vec{g}_I = \{\pm 1\}^{L\times 1}$. These vectors will be averaged using Proposition 3.

$$\vec{R}^T\vec{S} = \begin{bmatrix} \vec{g}_+ \\ \vec{g}_- \end{bmatrix}^T \begin{bmatrix} diag(\vec{r}_I) & diag(\vec{r}_R) \\ diag(\vec{r}_R) & -diag(\vec{r}_I) \end{bmatrix} \begin{bmatrix} Q_R - Q_I \\ Q_R + Q_I \end{bmatrix} \vec{b}_R$$

(39)

The orthogonal relation in (40) is derived from the fact that $G_R^T G_I = 0$ and $G_R^T G_R + G_I^T G_I = I$.

$$\begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix}^T \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} = I_{2L\times 2L}$$

(40)

The proof is shown in the Appendix D. Thus, the energy of the equivalent model is given by:

$$S^T S = \left\| \begin{bmatrix} Q_R - Q_I \\ -(Q_R + Q_I) \end{bmatrix} \vec{b}_R \right\|^2$$

$$S^T S = \left\| \begin{bmatrix} Z_- \\ Z_+ \end{bmatrix} \vec{1} \right\|^2$$

(41)

where $Z_+ = -(Q_R + Q_I) \cdot diag(\vec{b}_R)$ and $Z_- = (Q_R - Q_I) \cdot diag(\vec{b}_R)$ with both Z_+ and $Z_- \in \{\pm 1\}^{L\times U}$. These matrices can be treated independently as long as $Z_+ \neq Z_-$. For case where $Z_+ = Z_-$, the code matrix Q is real and the model becomes equivalent to the BPSK-code/BPSK-data. It is possible to reuse average likelihood for BPSK-code/QPSK-data CDMA in (30) and (31) by simply replacing D and Z with:

$$D = \begin{bmatrix} diag(\vec{r}_I) & diag(\vec{r}_R) \\ diag(\vec{r}_R) & -diag(\vec{r}_I) \end{bmatrix}$$

$$Z = \begin{bmatrix} Z_+ \\ Z_- \end{bmatrix}$$

(42)

Proposition 4 can be applied to the block vector $[\vec{g}_+, \vec{g}_-]^T$ in (39). Note that the redefining Z in (42) does not affect the parameter $P(Z)$. The new matrix $Z^T Z$ has L rows. The Frobenius Norm contains an additional term: $\tau^2(Q) = \tau^2(Z) + +\|Z_+^T Z_- - Z_-^T Z_+\|_F^2$.

$$P_{QPSK}(Z) \triangleq \frac{e^{-\beta\cdot\left(\tau^2(Z)+\left\|Z_+^T Z_- - Z_-^T Z_+\right\|_F^2\right)} w(Z)}{\sum_Y e^{-\beta\cdot\left(\tau^2(Y)+\left\|Y_+^T Y_- - Y_-^T Y_+\right\|_F^2\right)} w(Y)},$$

$$w(Z) = \begin{cases} 0 & rank(Z) \neq U \\ 1 & rank(Z) = U \end{cases}$$

(43)

The resulting likelihood is a compact form given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\vec{a}_+}\sum_{\vec{a}_-} \alpha(\beta,\vec{a}_+,\vec{a}_-)\, e^{-\gamma\|\vec{a}_+\|^2-\gamma\|\vec{a}_-\|^2}$$

$$\prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma}\,((\vec{r}_I)_i(\vec{a}_+)_i - (\vec{r}_R)_i(\vec{a}_-)_i)\right) \cosh\left(\sqrt{2\gamma}\,((\vec{r}_R)_i(\vec{a}_+)_i - (\vec{r}_I)_i(\vec{a}_-)_i)\right)$$

$$\vec{a}_+ = Z_+\,\vec{1}$$

$$\vec{a}_- = Z_-\,\vec{1}$$

(44)

where

$$\alpha(\beta,\vec{a}_+,\vec{a}_-) = \sum_t \sum_{(Z_+,Z_-)\in S_{\vec{a}_+,\vec{a}_-,t}} P(Z)$$

(45)

3.4.4 QPSK-Code/QPSK-Data Model

Finally, we can derive the average likelihood for the CDMA transmission model using QPSK and QPSK symbols. The model is given by:

$$\begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix} = \begin{bmatrix} G_R & -G_I \\ G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_R \\ \vec{b}_I \end{bmatrix} + \begin{bmatrix} \vec{n}_R \\ \vec{n}_I \end{bmatrix}$$

$$\begin{gathered} G_R = diag(\vec{g}_R) \\ G_I = diag(\vec{g}_I) \\ \vec{g}_R + \mathbb{i}\, \vec{g}_I \in \{\pm 1, \pm \mathbb{i}\}^{L\times 1} \\ \vec{b}_R + \mathbb{i}\, \vec{b}_I \in \{\pm 1, \pm \mathbb{i}\}^{U\times 1} \\ Q_R + \mathbb{i}\, Q_I \in \{\pm 1, \pm \mathbb{i}\}^{L\times U} \end{gathered} \tag{46}$$

The equivalent model provides a more convenient way of developing the average likelihood equation.

$$\vec{R} = \begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix}, \qquad \vec{S} = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_+ \\ -\vec{b}_- \end{bmatrix}, \qquad \vec{N} = \begin{bmatrix} \vec{n}_R + \vec{n}_I \\ -(\vec{n}_R - \vec{n}_I) \end{bmatrix}$$
$$\begin{gathered} \vec{R} = \vec{S} + \vec{N} \\ \begin{bmatrix} \vec{b}_+ \\ \vec{b}_- \end{bmatrix} = \begin{bmatrix} \vec{b}_R + \vec{b}_I \\ \vec{b}_R - \vec{b}_I \end{bmatrix} \end{gathered} \tag{47}$$

The correlator equation can be expressed in terms of new variables: $\vec{g}_+ = \vec{g}_R + \vec{g}_I$, $\vec{g}_- = \vec{g}_R - \vec{g}_I$, $\vec{b}_\Delta = -diag(\vec{b}_+)\, \vec{b}_-$, $Z_+ = Q_R \cdot diag(\vec{b}_+) + Q_I \cdot diag(\vec{b}_-)$, and $Z_- = Q_R \cdot diag(\vec{b}_+) - Q_I \cdot diag(\vec{b}_-)$.

$$\vec{R}^T \vec{S} = \begin{bmatrix} \vec{g}_+ \\ \vec{g}_- \end{bmatrix}^T \begin{bmatrix} diag(\vec{r}_I) & -diag(\vec{r}_R) \\ diag(\vec{r}_R) & diag(\vec{r}_I) \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & diag(\vec{b}_\Delta) \end{bmatrix} \begin{bmatrix} Z_- \\ Z_+ \end{bmatrix} \vec{1} \tag{48}$$

Then, the energy is given by:

$$\vec{S}^T \vec{S} = \left\| \begin{bmatrix} Z_- \\ Z_+ \end{bmatrix} \vec{1} \right\|^2 \tag{49}$$

with both Z_+ and $Z_- \in \{\pm 1\}^{L\times U}$.

The development of the likelihood is almost similar to the QPSK-code/BPSK-data except for the introduction of a new vector $\vec{b}_\Delta$. We begin our development by averaging over by applying Proposition 3.

$$\sum_{\vec{b}_+} P\left(\vec{b}_+\right) e^{\sqrt{2\gamma}\,(\vec{g}\ D\ F\ Z\ \vec{1}) - \gamma\,\|Z\ \vec{1}\|^2} = e^{\sqrt{2\gamma}\,(\vec{g}_R\ D\ F\ Z\ \vec{1}) - \gamma\,\|Z\ \vec{1}\|^2}$$

$$D = \begin{bmatrix} diag(\vec{r}_I) & -diag(\vec{r}_R) \\ diag(\vec{r}_R) & diag(\vec{r}_I) \end{bmatrix}$$

$$F = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & diag(\vec{b}_\Delta) \end{bmatrix}$$

$$Z = \begin{bmatrix} Z_+ \\ Z_- \end{bmatrix}$$

(50)

Then, we proceed to average over $\vec{g} = [\vec{g}_R; \vec{g}_I]$ is simplified by applying Proposition 5.

$$\sum_{\vec{g}_R} P(\vec{g}_R) e^{\sqrt{2\gamma}\,(\vec{g}_R\ D\ F\ Z\ \vec{1}) - \gamma\,\|Z\ \vec{1}\|^2} = e^{-\gamma\ \|Z\ \vec{1}\|^2} \prod_{i=0}^{2L-1} \cosh\left(\sqrt{2\gamma}\ \left(D\ F\ Z\ \vec{1}\ \right)_i\right)$$

(51)

The next step is averaging over $\vec{b}_\Delta$. This is given by:

$$\sum_{\vec{b}_\Delta} P\left(\vec{b}_\Delta\right) e^{-\gamma\,\|Z\ \vec{1}\|^2} \prod_{i=0}^{2L-1} \cosh\left(\sqrt{2\gamma}\ \left(D\ F\ Z\ \vec{1}\right)_i\right) =$$

$$\sum_{\vec{b}_\Delta} P\left(\vec{b}_\Delta\right) e^{-\gamma\,\|Z\ \vec{1}\|^2} \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma}\ \left((\vec{r}_I)_i\ \left(Z_+\vec{1}\right)_i - (\vec{r}_R)_i\ \left(\vec{b}_\Delta\right)_i \left(Z_-\vec{1}\right)_i\right)\right)$$

$$\cosh\left(\sqrt{2\gamma}\ \left((\vec{r}_R)_i\ \left(Z_+\ \vec{1}\right)_i + (\vec{r}_I)_i\ \left(\vec{b}_\Delta\right)_i \left(Z_-\vec{1}\right)_i\right)\right)$$

(52)

The product of cosine can be expressed as a sum of hyperbolic cosines as follows:

$$\cosh\left(\sqrt{2\gamma}\ \left((\vec{r}_I)_i\ \left(Z_+\vec{1}\right)_i - (\vec{r}_R)_i\ \left(\vec{b}_\Delta\right)_i\left(Z_-\vec{1}\right)_i\right)\right)$$

$$\cosh\left(\sqrt{2\gamma}\ \left((\vec{r}_R)_i\ \left(Z_+\ \vec{1}\right)_i + (\vec{r}_I)_i\ \left(\vec{b}_\Delta\right)_i\left(Z_-\vec{1}\right)_i\right)\right) =$$

$$\frac{1}{2}\cosh\left(\sqrt{2\gamma}\ \left((\vec{r}_R + \vec{r}_I)_i\ \left(Z_+\vec{1}\right)_i - \left(\vec{b}_\Delta\right)_i(\vec{r}_R - \vec{r}_I)_i\ \left(Z_-\vec{1}\right)_i\right)\right) +$$

$$\frac{1}{2}\cosh\left(\sqrt{2\gamma}\ \left((\vec{r}_R - \vec{r}_I)_i\ \left(Z_+\vec{1}\right)_i + \left(\vec{b}_\Delta\right)_i(\vec{r}_R + \vec{r}_I)_i\ \left(Z_-\vec{1}\right)_i\right)\right).$$

(53)

The equation is further simplified by applying the trigonometric identity in (54).

$$\sum_{d=\pm 1}\frac{1}{2}\cosh(a \pm d\cdot b) = \cosh(a)\cosh(b).$$

(54)

Finally, the average over Z_+ and Z_- is expressed in terms of the set $Z \in \mathcal{S}_{\vec{a}_+,\vec{a}_-,t}$. Note that the redefinition of Z in (46) does not affect the probability $P(Z)$ for the same reasoning that was already discussed in the previous section. See equation (43).

$$\lambda(R|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}}\sum_{\vec{a}_+}\sum_{\vec{a}_-}\alpha(\beta,\vec{a}_+,\vec{a}_-)\ e^{-\gamma\ \|\vec{a}_+\|^2-\gamma\ \|\vec{a}_-\|^2}.$$

$$\prod_{i=0}^{L-1}\left(\frac{1}{2}\cosh\left(\sqrt{2\gamma}\ ((\vec{r}_R + \vec{r}_I)_i\ (\vec{a}_+)_i)\right)\cosh\left(\sqrt{2\gamma}\ ((\vec{r}_R + \vec{r}_I)_i\ (\vec{a}_-)_i)\right) + \frac{1}{2}\cosh\left(\sqrt{2\gamma}\ ((\vec{r}_R - \vec{r}_I)_i\ (\vec{a}_+)_i)\right)\cosh\left(\sqrt{2\gamma}\ ((\vec{r}_R + \vec{r}_I)_i\ (\vec{a}_-)_i)\right)\right)$$

(55)

3.4.5 Summary of Average Likelihood Formulas for CDMA

The average likelihood for balanced CDMA signals is given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\vec{a}} \alpha(\beta,\vec{a}) e^{-\gamma\|\vec{a}\|^2} \prod_{i=0}^{L-1} \cosh(\sqrt{2\gamma}\,(D\,\vec{a})_i)$$

$$\vec{a} = Z\,\vec{1}$$

where

$$\alpha(\beta,\vec{a}) = \sum_t \sum_{Z\in S_{\vec{a},\tau}} P(Z).$$

Table 2 Average Likelihood Parameters

Code/Data	Matrix D	Matrix $Z \in S_{\vec{a},\tau}$
BPSK/BPSK	$D = diag(\vec{r}_R)$	$Z = Z_R$
BPSK/QPSK	$D = diag(\vec{r}_R + \vec{r}_I)$	$Z = Z_R$

For QPSK/BPSK and QPSK/QPSK data, the likelihood coefficients are given by:

$$\alpha(\beta,\vec{a}_R,\vec{a}_I) = \sum_t \sum_{(Z_+,Z_-)\in S_{\vec{a}_+,\vec{a}_-,t}} P(Z)$$

$$Z = \begin{bmatrix} Z_+ \\ Z_- \end{bmatrix}$$

(56)

the average likelihood equations for QPSK/BPSK:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\vec{a}_+}\sum_{\vec{a}_-} \alpha(\beta,\vec{a}_+,\vec{a}_-)\, e^{-\gamma\|\vec{a}_+\|^2-\gamma\|\vec{a}_-\|^2}$$

$$\prod_{i=0}^{L-1} \cosh(\sqrt{2\gamma}\,((\vec{r}_I)_i(\vec{a}_+)_i - (\vec{r}_R)_i(\vec{a}_-)_i))\cosh(\sqrt{2\gamma}\,((\vec{r}_R)_i(\vec{a}_+)_i - (\vec{r}_I)_i(\vec{a}_-)_i))$$

$$\vec{a}_+ = Z_+\vec{1}$$
$$\vec{a}_- = Z_-\vec{1}$$

(57)

and QPSK-code/QPSK-data:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{R},\vec{R}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\vec{a}_1}\sum_{\vec{a}_2} \alpha(\beta,\vec{a}_1,\vec{a}_2)\ e^{-\gamma\,\|\vec{a}_+\|^2-\gamma\,\|\vec{a}_-\|^2}$$

$$\prod_{i=0}^{L-1}\Big(\frac{1}{2}\cosh\big(\sqrt{2\gamma}(\vec{r}_R+\vec{r}_I)_i(\vec{a}_+)_i\big)\cosh\big(\sqrt{2\gamma}(\vec{r}_R-\vec{r}_I)_i(\vec{a}_-)_i\big) + \frac{1}{2}\cosh\big(\sqrt{2\gamma}(\vec{r}_R-\vec{r}_I)_i(\vec{a}_+)_i\big)\cosh\big(\sqrt{2\gamma}(\vec{r}_R+\vec{r}_I)_i(\vec{a}_-)_i\big)\Big)$$

$$\vec{a}_+ = Z_+\vec{1}$$
$$\vec{a}_- = Z_-\vec{1}$$

(58)

3.5 Unbalanced Load Classifier

The development of a Quasi-Average Likelihood function is similar for all four CDMA models. It is called quasi-average because the rule would require the estimation of the energy of each CDMA signature instead of an average. A strictly average likelihood over all possible signal-energies per user would result in a complex sum of product of error functions. The models require substituting:

$$\begin{bmatrix}\boldsymbol{I}+\Delta & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}+\Delta\end{bmatrix}\begin{bmatrix}\vec{b}_R+\vec{b}_I\\ \vec{b}_R-\vec{b}_I\end{bmatrix} \to \begin{bmatrix}diag(\vec{b}_R+\vec{b}_I) & \boldsymbol{0}\\ \boldsymbol{0} & diag(\vec{b}_R-\vec{b}_I)\end{bmatrix}\begin{bmatrix}1+\vec{\delta}\\ 1+\vec{\delta}\end{bmatrix}$$

(59)

where the energy per user E_i is given by:

$$E_i = \left(\sqrt{E_0}(1+\delta_i)\right)^2 \tag{60}$$

E_0 is a reference energy, $\Delta = \text{diag}(\vec{\delta})$ and $\vec{\delta}$ is a vector that contains the deviations in amplitude. In other words, the parameter $Z\,\vec{1}$ is substituted by $Z\,(\vec{1}+\vec{\delta})$ likelihood equation in (30) as shown next.

$$\sum_{All\ \vec{a}} \sum_{All\ \tau} \sum_{Z \in S_{\vec{a},\tau}} P(Z)\, e^{-\gamma \left\|\vec{a}+\vec{\xi}\right\|^2} \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma}\ \left(D\left(\vec{a}+\vec{\xi}\right)\right)_i\right)$$

$$\vec{a} = Z\,\vec{1}$$
$$\vec{\xi} = Z\,\vec{\delta} \tag{61}$$

3.6 Hard Decision Model

A simplification of (30) and (31) was proposed in [10]. The likelihood is computed for the case when the precision parameter approaches to infinity. The procedure is found in Appendix (B).

$$\lim_{\beta\to\infty} \alpha(\beta,\vec{a}) = \begin{cases} 0 & if\ \vec{a} \neq \vec{a}^{\star} \\ \dfrac{2^{-Count(a_i==0)}\left|S_{\vec{a},t_{min}}\right|}{\sum_{\vec{a}} 2^{-Count(a_i'==0)}\left|S_{\vec{a},t_{min}}\right|} & if\ \vec{a} = \vec{a}^{\star} \end{cases}. \tag{62}$$

$$\lim_{\beta\to\infty} \alpha(\beta,\vec{a}_+,\vec{a}_-) = \begin{cases} 0 & if\ (\vec{a}_+,\vec{a}_-) \neq (\vec{a}_+^{\star},\vec{a}_-^{\star}) \\ \dfrac{2^{-Count((\vec{a}_+)_i==0)}2^{-Count((\vec{a}_-)_i==0)}\left|S_{\vec{a}_+,\vec{a}_-,t_{min}}\right|}{\sum_{\vec{a}} 2^{-Count((\vec{a}_+)_i'==0)}2^{-Count((\vec{a}_-)_i'==0)}\left|S_{\vec{a}_+,\vec{a}_-,t_{min}}\right|} & if\ (\vec{a}_+,\vec{a}_-) = (\vec{a}_+^{\star},\vec{a}_-^{\star}) \end{cases}. \tag{63}$$

4. Status of the Research

During this second year of this effort, the research has concentrated in expanding the theory to cases of CDMA signals using BPSK and QPSK symbols. The next task will include simulations and the preparation of a conference paper. The research will move to estimation problems in CDMA using the average likelihood as shown in Table 3.

Table 3 In-House Schedule

	FY-13				FY-14				FY-15			
Tasks	**1QTR**	**2QTR**	**3QTR**	**4QTR**	**1QTR**	**2QTR**	**3QTR**	**4QTR**	**1QTR**	**2QTR**	**3QTR**	**4QTR**
Development of the Likelihood												
BPSK-BPSK Scheme												
Balanced Energy												
Simulations												
Documentation												
Development of the Likelihood												
BPSK-BPSK Scheme												
Balanced Energy												
Simulations												
Documentation												
Estimation												
Estimation of the Code												
Multiuser detection												
Simulations												
Documentation												

5. Conclusions

It is a known fact that average likelihood methods such as ALRT and MAP provide the best performance in noise. This means that any other classification method, i.e., neural network or ad-hoc cannot offer better performance in noise.

In the case of CDMA, computing the average likelihood function from its complex formulation in expression (10) turns quickly into an intractable problem. However, this report shows that it is possible to develop a simplified and compact the CDMA average likelihood for combinations of BPSK and QPSK symbols. This form consists of a sum of likelihood coefficients α and products of hyperbolic cosine functions. All the studied cases of CDMA transmission models using a combination of BPSK and QPSK symbols can be expressed in form similar to the previously studied BPSK-code/QPSK-data.

Expressing the likelihood of CDMA signals in a similar form facilitates the implementation of the algorithms and its validation. No significant issues are expected from these simulations, since the BPSK-code/BPSK-data has been successfully tested.

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7. Appendices

A. Conditional Likelihood for CDMA

In order to compute the conditional likelihood, we define the following parameters and variables:

Signal to noise ratio per chip	$\gamma_c = \frac{E}{N_0 L},$
Unnormalized signal	$\vec{s} = C \cdot \vec{b},$
CDMA transmit signal	$\vec{x} = \sqrt{E/L}\,\vec{s},$
Input to the correlator	$\vec{r} = \frac{\vec{y}}{\sqrt{N_0/2}}.$

Under the assumption of AWGN, the conditional likelihood can be expressed as a Gaussian multivariate distribution of a continuous vector $\vec{y}$. Then, we proceed to substitute them in the likelihood function:

$$\lambda(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{(\vec{y}-\vec{x})^H \Sigma^{-1} (\vec{y}-\vec{x})}{2}}$$

$$\lambda(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{\left(\sqrt{\frac{N_0}{2}}\vec{r} - \sqrt{E/L}\, C\vec{b}\right)^H \left(\sqrt{\frac{N_0}{2}}\vec{r} - \sqrt{E/L}\, C\vec{b}\right)}{2N_0/2}}$$

$$\lambda(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}) = \frac{1}{(\pi N_0)^{L/2}} e^{\frac{-\frac{N_0}{2}\vec{r}^H\vec{r} + \sqrt{\frac{E}{L}}\sqrt{\frac{N_0}{2}}\left(\vec{r}^H\, C\,\vec{b} + (C\,\vec{b})^H \vec{r}\,\right) - \frac{E}{L}(C\vec{b})^H C\vec{b}}{N_0}}$$

$$\lambda(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}) = \frac{1}{(\pi N_0)^{L/2}} e^{-\vec{r}^H\vec{r}/2 + \sqrt{\gamma/2}\,\left(\vec{r}^H\, C\,\vec{b} + (C\,\vec{b})^H \vec{r}\,\right) - \gamma (C\vec{b})^H C\vec{b}}$$

The final result is:

$$\lambda(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}) = \left(\frac{e^{-\langle \vec{r}, \vec{r} \rangle}}{(\pi N_0)^L}\right)^{1/2} e^{\sqrt{\gamma/2}\left(\vec{r}^H\, C\,\vec{b} + (C\,\vec{b})^H \vec{r}\,\right) - \gamma (C\vec{b})^H C\vec{b}}. \quad \square$$

B. Equivalent Models for CDMA

The standard transmission model for any CDMA transmission model

$$\vec{r} = G\,Q\,\vec{b} + \vec{n}$$

can be expressed in terms of real matrices as follows:

$$\begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix} = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_R \\ \vec{b}_I \end{bmatrix} + \begin{bmatrix} \vec{n}_R \\ \vec{n}_I \end{bmatrix},$$

where $Q = Q_R + \mathbb{i}\,Q_I$ is the code, $\vec{b} = \vec{b}_R + \mathbb{i}\,\vec{b}_I$ is the data, $G = G_R + \mathbb{i}G_I$ is a complex diagonal matrix that rotates the symbols in the complex plane, $\vec{r} = \vec{r}_R + \mathbb{i}\,\vec{r}_I$ is the correlator output at chip level or the input of to the correlator at frame level and $\vec{n} = \vec{n}_R + \mathbb{i}\,\vec{n}_I$ is the noise signal. Depending of the CDMA scheme, Q and $\vec{b}$ are either BPSK (real) or QPSK (complex) signals. In this report, we assume that QPSK symbols belong to the set $\{\pm 1, \pm\mathbb{i}\}$.

It was found that a development similar to the BPSK code/BPSK data scheme can be used if the model is substituted by:

$$\begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix} = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_R + \vec{b}_I \\ -(\vec{b}_R - \vec{b}_I) \end{bmatrix} + \begin{bmatrix} \vec{n}_R - \vec{n}_I \\ -(\vec{n}_R + \vec{n}_I) \end{bmatrix}.$$

The derivation is as can be simplified by noticing that:

$$\begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix} = \begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} = -\begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

Using these expressions, it is easy to compute:

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{b}_R \\ \vec{b}_I \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{n}_R \\ \vec{n}_I \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{r}_R \\ \vec{r}_I \end{bmatrix} = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_I \\ -\vec{b}_R \end{bmatrix} + \begin{bmatrix} \vec{n}_I \\ -\vec{n}_R \end{bmatrix}$$

and therefore,

$$\begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix} = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix} \begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \begin{bmatrix} \vec{b}_R + \vec{b}_I \\ -(\vec{b}_R - \vec{b}_I) \end{bmatrix} + \begin{bmatrix} \vec{n}_R - \vec{n}_I \\ -(\vec{n}_R + \vec{n}_I) \end{bmatrix}.$$

□

C. Key Propositions and Proofs

Proposition 1: The product of a diagonal matrix $D_{L\times L} = diag(\vec{d})$ and a vector $\vec{r}_{L\times 1}$ commutates as follows:

$$D\,\vec{r} = diag(\vec{r})\,\vec{d}$$

Proof:

$$D_{i,j} = d_i\,\delta_{i,j}$$

$$(D\,\vec{r})_i = \sum_i \left(d_i\,\delta_{i,j}\right) r_j$$

$$(D\,\vec{r})_i = \sum_i \left(\delta_{i,j}\,r_j\right) d_i$$

$$(D\,\vec{r})_i = \left(diag(\vec{r})\,\vec{d}\right)_i$$

□

Proposition 2: The product of the block matrix

$$G = \begin{bmatrix} G_R & G_I \\ -G_I & G_R \end{bmatrix}$$

with diagonal matrices $G_R = diag(\vec{g}_R)$ and $G_I = diag(\vec{g}_I)$

and a block vector $\vec{R}_{2L\times 1}$ with entries

$$\vec{R}_{2L\times 1} = \begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix}$$

commutates as follows:

$$G\ \vec{R} = \begin{bmatrix} diag(\vec{r}_I) & diag(\vec{r}_R) \\ -diag(\vec{r}_R) & diag(\vec{r}_I) \end{bmatrix} \begin{bmatrix} \vec{g}_R + \vec{g}_I \\ \vec{g}_R - \vec{g}_I \end{bmatrix}.$$

Proof:

$$\begin{bmatrix} G_R & -G_I \\ G_I & G_R \end{bmatrix} \begin{bmatrix} \vec{r}_R + \vec{r}_I \\ -(\vec{r}_R - \vec{r}_I) \end{bmatrix} =$$

$$\begin{bmatrix} (\vec{r}_R + \vec{r}_I)G_R - (\vec{r}_R - \vec{r}_I)G_I \\ (\vec{r}_R + \vec{r}_I)G_I - (\vec{r}_R - \vec{r}_I)G_R \end{bmatrix}$$

The desired result is obtained by applying Proposition 1:

$$\begin{bmatrix} diag(\vec{r}_R + \vec{r}_I)\vec{g}_R - diag(\vec{r}_R - \vec{r}_I)\vec{g}_I \\ diag(\vec{r}_R + \vec{r}_I)\vec{g}_I - diag(\vec{r}_R - \vec{r}_I)\vec{g}_R \end{bmatrix} =$$

$$\begin{bmatrix} diag(\vec{r}_I) & diag(\vec{r}_R) \\ -diag(\vec{r}_R) & diag(\vec{r}_I) \end{bmatrix} \begin{bmatrix} \vec{g}_R + \vec{g}_I \\ \vec{g}_R - \vec{g}_I \end{bmatrix}.$$

□

Proposition 3: The average of a function $f\left(C \cdot \vec{b}\right)$ over C and $\vec{b}$ where $C \in \{\pm 1\}^{L \times U}$ and $\vec{b} \in \{\pm 1\}^{U \times 1}$ is given by:

$$\frac{1}{2^{L \cdot U + U}} \sum_{C} \sum_{\vec{b}} f\left(C \cdot \vec{b}\right) = \frac{1}{2^{L \cdot U}} \sum_{C} f\left(C \cdot \vec{1}\right).$$

Proof:

We define $z_{i,j} = c_{i.j}\ b_i$, then

$$\frac{1}{2^{L \cdot U + U}} \sum_{C} \sum_{\vec{b}} f\left(C \cdot \vec{b}\right) = \frac{1}{2^{L \cdot U + U}} \sum_{Z} \sum_{\vec{b}} f\left(Z \cdot \vec{1}\right)$$

$$= \frac{1}{2^{L \cdot U}} \sum_{Z} f\left(Z \cdot \vec{1}\right) \sum_{\vec{b}} \frac{1}{2^{U}}.$$

The number of combinations of $\vec{b}$ is 2^U, therefore:

$$= \frac{1}{2^{L \cdot U}} \sum_{Z} f\left(Z \cdot \vec{1}\right).$$

□

Proposition 4: Let $\vec{b} = \vec{b}_R + \mathbb{i}\,\vec{b}_I$ with $\vec{b} \in \{-1, +1, -\mathbb{i}, \mathbb{i}\}^{U\times 1}$, $\vec{b}_R = Re\{\vec{b}\}$ and $\vec{b}_I = Im\{\vec{b}\}$. Also, let $\vec{b}_+ \in \{\pm 1\}^{U\times 1}$ and $\vec{b}_- \in \{\pm 1\}^{U\times 1}$, then

$$\frac{1}{2^{2\cdot U}}\sum_{\vec{b}_R}\sum_{\vec{b}_I} f\left(\vec{\alpha}^T\left(\vec{b}_R + \vec{b}_I\right)\right) f\left(\vec{\beta}^T\left(\vec{b}_R - \vec{b}_I\right)\right) = \frac{1}{2^U}\sum_{\vec{u}} f\left(2\vec{\alpha}^T\vec{b}_+\right)\frac{1}{2^U}\sum_{\vec{v}} f\left(2\vec{\beta}^T\vec{b}_-\right).$$

Proof:

The set of all possible combinations of $\vec{b}_R$ and $\vec{b}_I$ is shown in the next table.

$(\vec{\boldsymbol{b}}_{\boldsymbol{R}})_{\boldsymbol{i}}$	$(\vec{\boldsymbol{b}}_{\boldsymbol{I}})_{\boldsymbol{i}}$	$(\vec{\boldsymbol{b}}_+)_{\boldsymbol{i}} = (\vec{\boldsymbol{b}}_{\boldsymbol{R}} + \vec{\boldsymbol{b}}_{\boldsymbol{I}})_{\boldsymbol{i}}$	$(\vec{\boldsymbol{b}}_-)_{\boldsymbol{i}} = (\vec{\boldsymbol{b}}_{\boldsymbol{R}} - \vec{\boldsymbol{b}}_{\boldsymbol{I}})_{\boldsymbol{i}}$
+1	0	+1	+1
0	+1	+1	-1
-1	0	-1	-1
0	-1	-1	+1

When $\vec{b}_R + \vec{b}_I = 1$, $\vec{b}_R - \vec{b}_I$ can be either -1 or +1. Also, when $\vec{b}_R + \vec{b}_I = -1$, $\vec{b}_R - \vec{b}_I$ can be either -1 or +1. Therefore, for summation purposes, $\vec{b}_R + \vec{b}_I$ is statistically independent from $\vec{b}_R - \vec{b}_I$. In addition, the range of summation is in $\{\pm 1\}$.

$$\frac{1}{2^{2\cdot U}}\sum_{\vec{b}_R}\sum_{\vec{b}_I} f\left(\vec{\alpha}^T\left(\vec{b}_R + \vec{b}_I\right)\right) f\left(\vec{\beta}^T\left(\vec{b}_R - \vec{b}_I\right)\right) =$$

$$= \frac{1}{2^{2\cdot U}}\sum_{\vec{b}_+}\sum_{\vec{b}_-} f\left(\vec{\alpha}^T\vec{b}_+\right) f\left(\vec{\beta}^T\vec{b}_-\right)$$

$$= \frac{1}{2^U}\sum_{\vec{b}_+} f\left(\vec{\alpha}^T\vec{b}_+\right)\frac{1}{2^U}\sum_{\vec{b}_-} f\left(\vec{\beta}^T\vec{b}_-\right)$$

□

Proposition 5: Let $\vec{x}$ and $\vec{u}$ be vectors in $\mathbb{R}^{L\times 1}$. The average of the exponent of the dot product between $\vec{x}$ and $\vec{u}$ over the vector elements $x_i = \pm a_i$ is given by:

$$\frac{1}{2}\sum_{x_0=\pm a_0} \cdots \frac{1}{2}\sum_{x_{L-1}=\pm a_{L-1}} e^{\vec{u}^T\vec{x}} = \prod_{i=0}^{L-1}\cosh(u_i \cdot a_i).$$

Proof: This property can be proven by induction. Consider vectors $\vec{x}'$ and $\vec{u}'$ be vectors in $\mathbb{R}^{L+1\times 1}$. The dot product of vectors $\vec{x}'$ and $\vec{u}'$ can be constructed by multiplying by the exponential term $e^{u_L \cdot x_L}$ on both sides of the equation:

$$\frac{1}{2}\sum_{x_0=\pm a_0} \cdots \frac{1}{2}\sum_{x_{L-1}=\pm a_{L-1}} e^{\vec{u}^T\vec{x}+u_L\cdot x_L} = e^{u_L\cdot x_L}\prod_{i=0}^{L-1}\cosh(u_i \cdot a_i)$$

Then, averaging over the variable $x_L$results in:

$$\frac{1}{2}\sum_{x_0=\pm a_0} \cdots \frac{1}{2}\sum_{x_L=\pm a_L} e^{\vec{u}^T\vec{x}+u_L\cdot x_L} = \frac{1}{2}\sum_{x_L=\pm a_L} e^{u_L\cdot x_L}\prod_{i=0}^{L-1}\cosh(u_i \cdot a_i)$$

$$= \cosh(e^{u_L\cdot x_L})\prod_{i=0}^{L-1}\cosh(u_i \cdot a_i)$$

$$= \prod_{i=0}^{L}\cosh(u_i \cdot a_i)$$

Therefore, the formula applies to vectors in $\mathbb{R}^{L+1}$.

□

Proposition 6: The TSC is invariant to permutations and sign-inversions of rows and columns.

Proof: A permutation matrix $\mathcal{P}$ has the following property:

$$\mathcal{P}_1 \cdot \mathcal{P}_1^T = I \quad \text{and} \quad \mathcal{P}_2 \cdot \mathcal{P}_2^T = I$$

Using our definition of a TSC for a binary antipodal matrix,

$$\tau^2(C) = \|C^H C\|_F^2 - L \cdot U,$$

we have

$$\begin{aligned}
\tau^2(\mathcal{P}_1 C \mathcal{P}_2) &= \|\mathcal{P}_2^T C^T \mathcal{P}_1^T \mathcal{P}_1 C \mathcal{P}_2\|_F^2 - L \cdot U \\
&= \|C^T \mathcal{P}_1^T \mathcal{P}_1 C \mathcal{P}_2 \mathcal{P}_2^T\|_F^2 - L \cdot U \\
&= \|C^T C\|_F^2 - L \cdot U \\
&= \tau^2(C).
\end{aligned}$$

□

The following definitions are related to Proposition 7.

Definition: The set $\mathcal{S}_{\vec{a}}$ is the subset of real antipodal matrices $C = \{c_{i,j}\}^{L\times U}$ with:

$$a_i = \sum_{j=0}^{U-1} c_{i,j} \geq 0.$$

Definition: $\mathcal{S}_{\vec{a},t}$ is the subset of real antipodal matrices $C = \{c_{i,j}\}^{L\times U} \in \ \mathcal{S}_{\vec{a}}$ with:

$$\tau^2(C) = t.$$

Definition: $\mathcal{S}_{\vec{a}_+,\vec{a}_-,t}$ is the subset of pair of matrices (Z_+, Z_-) with $Z_+ = \{\pm 1\}^{L\times U}$ and $Z_- = \{\pm 1\}^{L\times U}$ with:

$$(\vec{a}_+)_i = \sum_{j=0}^{U-1} (Z_+)_{i,j} \geq 0$$

$$(\vec{a}_-)_i = \sum_{j=0}^{U-1} (Z_-)_{i,j} \geq 0$$

$$\tau^2\left(\frac{(Z_+ + Z_-)}{2} + \mathbb{i}\,\frac{(Z_+ - Z_-)}{2}\right) = t.$$

Proposition 7. The limit of $\alpha_{L,U,\vec{a}}(\beta)$as β approaches infinity depends on the feature vectors as follows:

$$\lim_{\beta\to\infty} \alpha_{L,U,\vec{a}}(\beta) = \begin{cases} 0 \;\; for \; \vec{a} \neq \vec{a}^{\star} \\ \dfrac{\frac{2^L}{2^{Sum(a_i=0)}} \left| \mathcal{S}_{\vec{a},t_{min}} \right|}{\sum_{\vec{a}} \frac{2^L}{2^{Sum(a_i=0)}} \left| \mathcal{S}_{\vec{a},t_{min}} \right|} > 0 \;\; for \; \vec{a} = \vec{a}^{\star} \end{cases}$$

Proof: The expression can be converted in a polynomial rational function by substituting $e^{-\beta} \to y$ and divide by the exponent with the minimum Total Square Correlation $y^{\tau_{min}}$ in both side of the equation.

$$\alpha_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Count(a_i=0)}} \sum_t \sum_{Z\in \mathcal{S}_{\vec{a},t}} y^t}{\sum_{\vec{a}} \frac{2^L}{2^{Count(a_i=0)}} \sum_t \sum_{Z\in \mathcal{S}_{\vec{a},t}} y^t}$$

$$= \frac{\frac{2^L}{2^{Count(a_i=0)}} \sum_t \sum_{Z\in \mathcal{S}_{\vec{a},\tau}} y^{t-t_{min}}}{\sum_{\vec{a}} \frac{2^L}{2^{Count(a_i=0)}} \sum_t \sum_{Z\in \mathcal{S}_{\vec{a},t}} y^{t-t_{min}}}$$

$$t = \tau(Z) \geq t_{min} \geq 0$$

$$\vec{a} = abs(Z \cdot \vec{1})$$

Since $t - t_{min} \geq 0$, taking the limit:

$$\lim_{y\to 0} y^{t-t_{min}} = \begin{cases} 0 \; for \; t \neq t_{min} \\ 1 \; for \; t = t_{min} \end{cases}$$

eliminates all terms with Total Square Correlation terms greater than τ_{min}. The result is:

$$\alpha_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Count(a_i=0)}} \sum_{Z\in \mathcal{S}_{\vec{a},t_{min}}} 1}{\sum_{\vec{a}} \frac{2^L}{2^{Count(a_i=0)}} \sum_{Z\in \mathcal{S}_{\vec{a},t_{min}}} 1} \geq 0$$

The summation over all the terms in the set is just the cardinality of the set, therefore:

$$\alpha^{\star}_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Count(a_i=0)}} \left| \mathcal{S}_{\vec{a},t_{min}} \right|}{\sum_{\vec{a}} \frac{2^L}{2^{Count(a_i=0)}} \left| \mathcal{S}_{\vec{a},t_{min}} \right|}.$$

$\square$

D. Orthogonal Relationships

Let $G = G_R + G_I$ be a diagonal matrix with diagonal elements given by

$$G = diag(\vec{g}_R + \mathbb{i}\,\vec{g}_I)$$

and $\vec{g}_R + \mathbb{i}\,\vec{g}_I \in \{\pm 1, \pm \mathbb{i}\}^{L\times 1}$.

Then the following equations are true:

$$G_R^T G_I = diag(\vec{g}_R)\, diag(\vec{g}_I) = \mathbf{0}$$

$$G_R^T G_R + G_I^T G_I = I.$$

Proof:

Whenever $\vec{g}$ takes a real value, the imaginary part is zero by definition.

$$(\vec{g}_R + \mathbb{i}\,\vec{g}_I)_i \in \mathbb{R}$$

$$Im\{(\vec{g}_R + \mathbb{i}\,\vec{g}_I)_i\} = 0$$

$$(\vec{g}_I)_i = 0$$

Whenever $\vec{g}$ takes a imaginary value, the real part is zero by definition.

$$(\vec{g}_R + \mathbb{i}\,\vec{g}_I)_i \in Imaginary$$

$$Re\{(\vec{g}_R + \mathbb{i}\,\vec{g}_I)_i\} = 0$$

$$(\vec{g}_R)_i = 0$$

Therefore, the product $G_R^T G_I$ is a zero matrix.

$$G_R^T G_I = diag(\vec{g}_R) \cdot diag(\vec{g}_I)$$

$$= diag(\{\, (\vec{g}_R)_i (\vec{g}_I)_i \,\}^{L\times 1})$$

$$= diag(\{\, 0 \,\}^{L\times 1})$$

$$= \mathbf{0}.$$

By the definition of $\vec{g}_R$ and $\vec{g}_I$, we have that:

$$\vec{g}_R + \vec{g}_I = \pm 1.$$

Using the previous results, we have that the identity is formed by $G_R^T G_R + G_I^T G_I$.

$$I = diag(\vec{g}_R + \vec{g}_I) \cdot diag(\vec{g}_R + \vec{g}_I)$$

$$I = diag(\vec{g}_R + \vec{g}_I) \cdot diag(\vec{g}_R + \vec{g}_I)$$

$$I = diag(\vec{g}_R)^T \cdot diag(\vec{g}_R) + diag(\vec{g}_R)^T \cdot diag(\vec{g}_I)$$
$$+ diag(\vec{g}_I)^T \cdot diag(\vec{g}_R) + diag(\vec{g}_I)^T \cdot diag(\vec{g}_I)$$

$$I = diag(\vec{g}_R)^T \cdot diag(\vec{g}_R) + diag(\vec{g}_I)^T \cdot diag(\vec{g}_I)$$

$$I = G_R^T G_R + G_I^T G_I$$

$\square$

8. Acronyms

ALRT	Average Likelihood Ratio Test
CDMA	Code Division Multiple Access
BPSK	Binary Phase Shift Keying
QPSK	Quadrature Phase Shift Keying
LRT	Likelihood Ratio Test
MAP	Maximum A Posteriori Probability
TSC	Total Squared Correlation
$\tau^2()$	Total Squared Correlation function
γ	Chip Signal-to-Noise Ratio
E	Energy per symbol per user
$\lambda(\vec{r}\|\mathcal{H})$	Likelihood function
$\lambda(\vec{r}\|\mathcal{H}, C, \vec{b})$	Conditional likelihood function
C	Spreading matrix
$C^{L\times U}$, $C_{L\times U}$	Matrix with specified rows (L) and columns (U)
$\vec{c}_{*,i}$	Column vector i of matrix C
$(\quad)^H$	Hermitian operator, i.e., the complex conjugate transpose
C_R	Real part of the matrix C
C_I	Imaginary part of the matrix C
$C^{\star}$	TSC optimal spreading matrix
$\vec{b}$	Information vector
$\vec{b}_R$	Real part of the vector $\vec{b}$
$\vec{b}_I$	Imaginary part of the vector $\vec{b}$
$(\vec{g})_i$	Element i of vector $\vec{g}$
$\alpha_{L,U,\vec{a}}$	Coefficients of the CDMA likelihood function
$\alpha^{\star}$	Non-vanishing alpha coefficients
$\vec{a}^{\star}$	Feature vector for CDMA detection
$\sum_{\vec{a}} f(\vec{a})$	Summation over all possible vector coefficients

$\sum_{C} f(C)$	Summation over all possible matrix coefficients
$\boldsymbol{I}$	Identity matrix
$\mathbf{0}$	Zero matrix
$\vec{1}$	Row Column vector with elements equal to 1
$diag(\vec{g})$	Diagonal matrix with elements $(\vec{g})_i$ in the diagonal.
$\mathbb{E}\{\quad\}$	Expectation operator
$\mathbb{R}$	Set of real numbers
$\lvert S \rvert$	Cardinality of a set S, i.e., the total number of elements in the set
$\{\pm 1\}$	Set of antipodal symbols
$\{\pm 1, \pm \mathbb{i}\}$	Set of QPSK symbols